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Uniqueness Theorems Determined by Function Values at the Roots of Unity

CHIN-HUNG CHING

Department of Mathematics, University of Melbourne, Parkville, Victoria, Australia

AND

CHARLES K. CHUI

Department of Mathematics, Texas A & M University, College Station, Texas 77843

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1. INTRODUCTION

For each positive integer N, let $w_N{}^k = \exp(i2\pi k/N)$, $k = 1,..., N$, be the Nth roots of unity. If f is a continuous function on the unit circle T: $|z| = 1$, satisfying

$$\sum_{k=1}^{N} f(w_N{}^k) = 0, \tag{1}$$

for all $N = 1, 2,...$, it is natural to ask if f is the zero function. However, it is clear that any function f defined by

$$f(z) = \sum_{k=1}^{\infty} a_k(z^k - z^{-k}),$$

where the series converges on T, always satisfies (1) for all $N = 1, 2,...$, though f may not be the zero function. Hence, we will only consider functions holomorphic in the open unit disc. In this paper, we obtain the following results.

THEOREM 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *where*

$$a_n = O(1/n^{1+\epsilon}) \tag{2}$$

for some $\epsilon > 0$, *satisfy* (1) *for every* $N = 1, 2,...$. *Then* f *is the zero function.*

Condition (2) is only a sufficient one where we put restriction on the asymptotic behavior of the coefficients a_n. We have another result where we assume a global condition on the a_n.

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THEOREM 2. *Let* $f(z) = \sum_{k=0}^{\infty} a_n z^n$, *where*

$$\sum_{k=N}^{\infty} | a_n | = o(1/N), \tag{3}$$

satisfy (1) *for all* $N = 1, 2, \ldots$. *Then* f *is the zero function.*

We have a sharper result for some gap series.

THEOREM 3. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{q^k},$$

where q *is a positive integer and* $\sum_{k=0}^{\infty} | a_k | < \infty$. *If* f *satisfies* (1) *for all* $N = 1, 2, \ldots$, *then* f *is the zero function.*

We also remark that none of the conditions in (1) can be omitted as in the following theorem.

THEOREM 4. *Let* N *be a positive integer. There exists a unique polymonial* p_N *of degree* N, *leading coefficient equal to one and* $p_N(0) = 0$ *such that* p_N *satisfies* (1) *for all positive integers* n *different from* N.

2. PROOFS OF THE THEOREMS

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} | a_n | < \infty$. Suppose that f satisfies (1) for all $N = 1, 2, \ldots$. We first note that

$$a_0 = 1/2\pi i \int_{|z|=1} (f(z)/z)\, dz$$

$$= \lim_{n\to\infty} 1/n \sum_{k=1}^{n} f(w_n{}^k) = 0.$$

For each $N \geqslant 1$, it is clear that

$$1/N \sum_{k=1}^{N} w_N^{kn} = \begin{cases} 0 & \text{if } N \nmid n \\ 1 & \text{if } N \mid n. \end{cases}$$

Here, $N \mid n$ means, as usual, that N is a factor of n. By the absolute convergence of the infinite series of the coefficients a_n, we obtain

$$1/N \sum_{k=1}^{N} f(w_N{}^k) = \sum_{n=0}^{\infty} a_n(1/N) \sum_{k=1} w_N^{kn}$$

$$= \sum_{n=0}^{\infty} a_{nN} = \sum_{n=1}^{\infty} a_{nN}.$$

By (1), we have a system of equations

$$A_N = \sum_{k=1}^{\infty} a_{kN} = 0, \tag{4}$$

where $N = 1, 2,\dots$. To solve these equations, we make use of the Möbius function $\mu(n)$,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k, \\ 0 & \text{if } p^2 \mid n \text{ for some } p > 1, \end{cases}$$

where $p_1 ,\dots, p_k$ are distinct primes. It is well known (cf. [1, Theorem 263]) that

$$\sum_{k \mid n} \mu(k) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{5}$$

Now,

$$\sum_{n=1}^{N} \mu(n) A_n = \sum_{n=1}^{N} \mu(n) \sum_{k=1}^{\infty} a_{kn}$$

$$= \sum_{n=1}^{N} \mu(n) \sum_{n \mid j}^{j \geqslant n} a_j$$

$$= \sum_{j=1}^{N} a_j \sum_{n \mid j} \mu(n) + \sum_{j=N+1}^{\infty} \sum_{n \mid j}^{1 \leqslant n \leqslant N} a_j \mu(n).$$

By (4) and (5), we have for all $N = 1, 2,\dots,$

$$a_1 + \sum_{j=N+1}^{\infty} \Big(\sum_{n \mid j}^{1 \leqslant n \leqslant N} \mu(n) \Big) a_j = 0. \tag{6}$$

Let $d(m)$ denote the number of divisors of m. Then

$$d(m) = O(m^\delta) \tag{7}$$

for all positive δ (cf. [1, Theorem 315]).

Suppose that the condition (2) is satisfied for some $\epsilon > 0$. Then by (6) and (7), we have, choosing $\delta = \epsilon/2$,

$$| a_1 | \leqslant \sum_{j=N+1}^{\infty} \Big(\sum_{n|j} |\mu(n)|\Big) | a_j |$$

$$\leqslant \sum_{j=N+1}^{\infty} d(j) | a_j | = O(1/N^{\epsilon/2}).$$

Hence, $a_1 = 0$. For each fixed $k \geqslant 1$, let $b_j = a_{jk}$. By (2) and (4) we have

$$\sum_{s=1}^{\infty} b_{js} = 0$$

for all j and

$$b_j = O(1/(j)^{1+\epsilon}).$$

By the same argument as above, we conclude that $a_k = b_1 = 0$. This completes the proof of Theorem 1.

Actually, in the above proof, we only need the convergence of the series

$$\sum_{n=1}^{\infty} d(n) | a_{nk} | \tag{8}$$

for all k instead of the condition (2). However, the behavior of $d(n)$ is quite irregular, namely, $\liminf d(n) = 2$ and the true "maximum order" of $d(n)$ is about $2^{\log n/\log\log n}$, so that in general it is rather difficult to determine the convergence of the series in (8).

To prove Theorem 2, we again use (6) and obtain

$$| a_1 | \leqslant N \sum_{m=N+1}^{\infty} | a_m |.$$

In general, for $k \geqslant 1$, we use the same argument as in the proof of Theorem 1 to obtain

$$| a_k | \leqslant N \sum_{m=N+1}^{\infty} | a_{mk} |.$$

The proof of Theorem 2 is completed by using the hypothesis (3).

Let $q > 1$. By (6) and similar argument as above, we obtain

$$\begin{aligned} | a_k | &\leqslant \sum_{m=N+1}^{\infty} \left| \sum_{j|q^m} \mu(j) \right| | a_{m+k} | \\ &= \sum_{m=N+1} \left| \sum_{j|q} \mu(j) \right| | a_{m+k} | \\ &\leqslant d(q) \sum_{m=N+1}^{\infty} | a_{m+k} | \leqslant d(q) \sum_{m=N+1}^{\infty} | a_m | \end{aligned}$$

for each $k = 1, 2,\ldots$. Hence, by letting N tend to infinity, we have $a_1 = a_2 = \cdots = 0$, completing the proof of Theorem 3.

To prove Theorem 4, we write

$$p_N(z) = z^N - (c_{N-1}z^{N-1} + \cdots + c_1 z).$$

Then the conditions in (1) are trivially satisfied by p_N for all $n > N$. Hence, to determine the coefficients $c_1, \ldots, c_{N-1}$, so that p_N satisfies (1) for all $n < N$ we need only consider the following system of $N - 1$ linear equations

$$\begin{aligned} c_1 + \cdots + c_{N-1} &= r_1 \\ c_2 + c_4 + \cdots &= r_2 \\ \cdots \\ c_{N-1} &= r_{N-1}, \end{aligned}$$

where

$$r_k = \begin{cases} 1 & \text{if } k \mid N, \\ 0 & \text{otherwise.} \end{cases}$$

Since the matrix of coefficients of the c_j is an upper triangular matrix with determinant equal to one, there is a unique solution for the above system of equations.

Reference

1. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Oxford Univ. Press, London/New York, 1954.